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A CONSTRUCTION OF 3-CONNECTED GRAPHS

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ABSTRACT

We show that the 3-connected graphs can be generated from the complete graph on four vertices and the complete 3,3 bipartite graph by adding vertices and adding edges with endpoints on two edges meeting at a 3-valent vertex.

1. Introduction

In this paper we present a new recursive construction of the 3-connected graphs. Several recursive constructions are known. Tutte [3] has proved that all 3-connected graphs can be generated from the wheels by repeated application of adding edges between existing vertices, and splitting vertices so as not to produce triangles. Another well known generation procedure (see, for example, [1]) generates the 3-connected graphs from the complete graph on four vertices by adding edges, where an added edge can join two existing vertices or can meet the relative interiors of edges, creating new vertices.

In this paper we show that the 3-connected graphs can be generated from the complete graph on four vertices and the complete bipartite graph $K_{3,3}$ by repeated application of two processes: Adding a vertex, and adding an edge with endpoints on two edges meeting at a 3-valent vertex.

2. Definitions and notation

The graphs in this paper are without loops or multiple edges. If P is a path in a graph, and if x and y are two vertices of P, then $P[x,y]$ is the portion of P

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joining x and y, and $P(x,y)$ is $P[x,y]$ with the endpoints (but not the end edges) removed (such a path will be called an **open path**, or **open arc**). If we wish to indicate a subpath with one end included and one excluded we use notation such as $P[x,y)$. An **arc** of a graph G is a maximal path $P[x,y]$ such that all vertices of $P(x,y)$ are 2-valent in G. The vertices of a graph of valence at least three will be called **principle** vertices. A graph H is a **refinement** of a graph G if H is obtained from G by adding vertices to the edges of G. If H is a refinement of G, and all vertices of G are principle vertices, then we say that G is the **underlying graph** of H, and we denote it by H^*. For any given graph G, we will label the vertices of G^* with the same labels as the corresponding principle vertices in G. An edge with vertices x and y will be denoted xy. A path formed by joining smaller paths A and B at a common endpoint will be denoted AB. The same notation will be used to join more than two paths, including paths consisting of single edges, thus a path that traverses A then xy then B would be denoted $AxyB$, and so on.

A graph G is 3-connected provided between any two vertices x and y there are three paths that meet only at x and y. If G is a graph such that G^* is 3-connected, we say that G is **3*-connected**. We say that a graph G is obtained from H by **adding vertex** v, provided we obtain the graph H by removing from G, the vertex v and the edges meeting it, and then coalescing any pairs of edges meeting at resulting 2-valent vertices, into single edges. Such a vertex v is called a **removable** vertex of G.

The proof of the following lemma is a routine application of the definition of 3-connectivity.

LEMMA 1: *If H is 3-connected, G is obtained from H by adding v, and the edges of G meeting v do not all meet the same edge of H (after the removal of v and coalescing of edges), then G is 3-connected.*

If H is a subgraph of G and v is a vertex in G but not in H, then a path **from** v **to** H is a path $P[v,x]$ with $P \cap H = x$. A family of paths from v to H is **independent** provided no two paths have a vertex in common, other than v.

LEMMA 2: *If G is 3-connected, H is a subgraph of G containing at least three vertices, and v is a vertex of G not in H, then there are three independent paths in G from v to H.*

Proof: We add a new vertex w to G by attaching it to three vertices of H. By

Lemma 1, the new graph G' is 3-connected. We take three paths from v to w in G', meeting only at v and w. Clearly a subpath of each of these paths joins v to H. ∎

A graph H is obtained from a graph G by adding an arc, provided H is obtained by adding a path $P[x, y]$ to G, such that P meets G only at x and y, and x and y do not lie on the same arc of G. The vertices x and y can be vertices of G or can be relative interior points of edges of G. Another easy application of the definition of 3-connectivity shows:

LEMMA 3: *If G is 3^*-connected and H is obtained from G by adding arcs, then H is 3^*-connected.*

We shall denote the complete graph on four vertices by C_4, and the complete bipartite graph on two sets of three vertices by $K_{3,3}$. The following is a well known property of 3-connected graphs (see, for example, [1], Lemma 4):

LEMMA 4: *Every 3-connected graph contains a refinement of C_4.*

A **triangle** in a graph is a subgraph consisting of three edges xy, yz and zx. The vertices x, y and z are called the **vertices of the triangle**, and the triangle will be denoted xyz. If G has a triangle xyz with at least one vertex 3-valent in G, then xyz will be called a t-**triangle** of G.

A **wheel** is a graph consisting of a simple circuit C, and a vertex h not on C, that is joined to each vertex of C. The edges meeting h are called the **spokes** of the wheel.

3. The main theorem

THEOREM 1: *If G is a 3-connected graph, other than C_4 or $K_{3,3}$, then G has either a removable vertex or a t-triangle.*

Proof: We assume that G does not have a t-triangle. The first part of our proof involves an inductive construction of a subgraph H of G containing all of the vertices of G, and such that H^* has a removable vertex.

We shall do this by constructing a sequence of subgraphs $G_0, \dots G_k$, of G, such that $G_k = H$. We begin with a subgraph G_0 of G that is a refinement of C_4. We now describe how to construct G_n from G_{n-1}. The construction assumes the existence of a vertex v of G that is not in G_{n-1}. (The case where every refinement of C_4 contains all vertices of G will be treated last.)

We begin by adding arcs that join v, to G_{n-1}, creating a graph G'_{n-1}. We do not specify how many arcs we add. It can vary from zero to the maximum number possible for the particular G_{n-1}. (Different choices generally give different sequences of G_i's, and later on we will be considereing the set of all possible sequences.)

Since G is 3-connected, there are three independent paths from v to G'_{n-1}. If the three paths do not all end on the same arc of G'_{n-1}, then we add the three arcs to G'_{n-1}, creating G_n. By Lemmas 3 and 4, G_n is 3*-connected if G_{n-1} is.

Suppose that all choices of the three independent paths from v to G'_{n-1} end on the same arc, A of G'_{n-1}. Let U be the union of all paths in G from v to G'_{n-1}. As we traverse A from one endpoint to the other, let x be the first vertex of U, and y be the last vertex of U encountered.

Since there is a path from v to x missing A, and a path from v to y missing A, there is a vertex v' on the intersection of these paths such that there are two independent paths P_1 and P_2 in U, from v' to x and y, missing A (see Fig. 1).

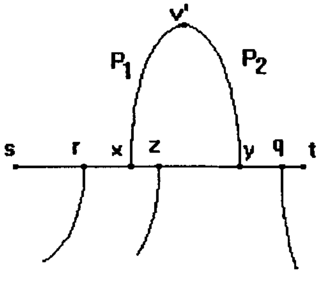

Fig. 1

Note that if there is a path P_3 from any vertex of U to any arc of G'_{n-1} other than A, then one can either extend that path to intersect P_1 or P_2, or take a subpath of it, so that there are three independent paths from some vertex of U to G'_{n-1}, not all ending on A, thus we can assume that no such path P_3 exists.

Case 1: Some path $P(z, w)$ joins $A(x, y)$ to some open arc of G'_{n-1}, other than A, with z on $A(x, y)$ (see Fig. 1). We replace $A(x, y)$ in G'_{n-1}, with P_1P_2. Note that this produces a graph G''_{n-1} homeomorphic to G'_{n-1}. Now we add the vertex z and the paths: $A(x, z]$, $A[z, y)$, and P, to G''_{n-1}, to produce the graph G_n.

Case 2: No such path exists. Let the endpoints of A be s and t, and let the order on A be s, x, y, t. Since G is 3-connected, there must be a path in G joining some arc of G'_{n-1}, other than A, to either $A(s,x]$ or $A[y,t)$, otherwise we can separate G at s and t. Suppose such a path P' is joined to $A(s,x]$, and among all such paths let P' be one whose endpoint r on $A(s,x]$ is closest to x. If no such path exists, we let $r = s$. Suppose that such a path P'' is joined to $A[y,t)$, and that among all such paths, P'' is one with an endpoint q nearest to y. If no such path exists, we let $q = t$. There must be a path in G from $A[s,r)$ to $A(r,t]$, or from $A[s,q)$ to $A(q,t]$, for otherwise we could disconnect G by removing r and q. Suppose, without loss of generality, that we have a path $Q[a,b]$, with a on $A[s,r)$, from $A[s,r)$ to $A(r,t]$. In G'_{n-1}, we replace $A(a,b)$ with Q, producing a graph G''_{n-1} homeomorphic to G'_{n-1}. Now, to G''_{n-1} we add $A[a,r]$, $A[r,b]$ and P', to produce G_n.

We continue adding arcs, vertices and paths in this manner. The process stops when we reach a subgraph G_k of G, that is 3*-connected and contains all vertices of G. Furthermore, G_k^* has a removable vertex v (corresponding to the principle vertex added to a subgraph homeomorphic to G'_{k-1}).

The vertices added at each step (from which the three independent paths emanate) will be called **special vertices** of G. Now, among all subgraphs G_k that can be constructed in this manner, we let H be one with the maximum number $m(H)$, of special vertices, and we also assume that among all with $m(H)$ special vertices, H is one such that the sum $j(H)$ of the lengths of the arcs from v to the subgraph homeomorphic to G'_{k-1} is minimal. We shall now refer to the subgraph homeomorphic to G'_{k-1}, to which v and its arcs were added, as K.

The next part of the proof consists in showing that each of the arcs from v to K is an edge. Suppose that $A[v,v']$ is one of these arcs and that x is a vertex on the open arc A. We treat four cases.

Case 1: In G, x is joined to a vertex y on an arc $B[v,z]$, with v' and z joined by an arc C in H. We replace C by the path $A[v',x]xyB[y,z]$, producing a subgraph of G homeomorphic to H, such that the sum of the lengths of the arcs meeting v is less than $j(H)$.

Case 2: In G, x is joined to a vertex y on an arc $B[v,z]$ such that no arc in H joins v' and z. By Lemma 3, adding the path $B[z,y]yxA[x,v']$ to K creates a new subgraph L of G, that is 3*-connected. If we add v, the paths $A[v,x]$, $B[v,y]$ and the other arc of H meeting v, to L, we obtain a graph H' consisting of H with the edge xy added. The graph H' is 3*-connected and v is removable in $(H')^*$, but $j(H')$ is smaller than $j(H)$.

Case 3: In G, x is joined to a vertex y on an arc C, not meeting v, and y and v' are joined by an arc. An argument similar to Case 1 shows that $j(H)$ is not minimal.

Case 4: In G, x is joined to a vertex y on an arc C, not meeting v, and y and v' are not joined by an arc in H. An argument similar to Case 2 shows that $j(H)$ is not minimal.

We conclude that in H, each edge meeting v also meets K. This implies that each edge in G meeting v meets K, for if an edge pv did not meet K, then p would not be in H, contradicting the fact that H contains all vertices of G.

The next step of our proof involves adding edges of $G - K$ missing v, to K, one at a time, while preserving the 3*-connectivity of the resulting graphs. Suppose that we have added all edges that we can that preserve 3*-connectivity, producing a graph K'. Now, any edge that we add creates a double edge in the underlying graph, thus every edge to be added has both vertices on one arc of K' (such edges will now be refered to as **chords**). Suppose that $A[x, y]$ is such an arc. Note that at least one edge from v must meet $A(x, y)$, by the 3-connectivity of G.

We shall show that all edges missing v can be added by showing that if this were not true, then we can find another subgraph H', such that $m(H') = m(H)$ but a vertex of A does not lie in H', thus we can construct a graph H'' from H' with $m(H'') > m(H)$.

First we note that if vv' is an edge of G meeting $A(x, y)$, but is not an edge of H, we may substitute vv' for any edge of H meeting v and $A(x, y)$, and obtain a graph with the same value of m. Next we note that if st is a chord, then there must be a vertex between s and t on A to prevent double edges in G. If the edges of H, from v to A all meet $A[x, s]$ or all meet $A[t, y]$ (we shall assume an order of x, s, t, y on $A[x, y]$), then we can replace $A(s, t)$ in H by st, and obtain the desired graph H' missing all vertices between s and t. It now follows that if two edges from v in G meet either $A[x, s]$ or $A[t, y]$, then we can reach a contradiction. We shall now treat three cases.

Case 1: There are three edges vp, vq and vr in G from v to A. In all cases the graph H' is easily constructed. We shall show the case where p is on $A(x, s)$, and q and r are on $A(s, t)$. The other cases are similar. We shall assume an ordering of x, p, s, q, r, t, y on A. In K', we replace $A(s,t)$ by st. Then we add v, vp (and the path $vqA(q,s)$, if two edges of H meet v and A), and the edges of H meeting v and missing A. This new graph H' now missed the vertex r.

Case 2: There do not exist three edges from v to A in G and there exists only one edge from v in G that misses A. In this case v is 3-valent in G. Let vp and vq be the two edges to A, with the same ordering as before. There must be a vertex on $A(p,q)$, for otherwise vpq is a t-triangle in G. Let st be a chord meeting $A(p,q)$. If all such chords had both vertices on $A(p,q)$, then p and q would disconnect G, thus we may assume that t is on $A(p,q)$ and s (by symmetry) is on $A[x,p)$.

If there is a vertex w on $A(p,t)$ then we can construct H' by replacing $A(s,t)$ by st in K', then adding $vpA(p,s)$, vq and the third edge of H meeting v. Now H' misses w. A similar argument holds if there is a vertex on $A(s,p)$. It now follows that spt is a t-triangle in G unless a chord meets p. Let pw be such a chord. The graph H' is now easily constructed. Again we shall show one case, the others are similar.

Suppose that w is on $A(q,y]$. We replace $A(p,w)$ by pw, in K'. We add v, vp, $vqA(q,w)$ and the third edge meeting v in H. This new graph misses t.

Case 3: There are at least two edges in G from v missing A, that do not both meet another arc meeting x or y. In this case we add these two edges and one edge vp from v to A, to K', and we shall now refer to this graph as H. We now note that every chord has p between its vertices on A, for otherwise as we have observed above we can easily construct the graph H' missing a vertex between the vertices of the chord. Let st be such a chord, with the ordering on A being x, s, p, t, y.

If there is a vertex w on $A(s,p)$ then we replace $A(s,t)$ with st, in K'. We add v, $vpA(p,t)$, and the other two edges of H meeting v, creating a graph H' missing w. Similarly, there cannot be a vertex on $A(p,t)$, thus spt is a t-triangle in G unless a chord meets p. Suppose we have a chord pw. If w is on $A[x,s)$, we replace $A(p,w)$ with pw in K', then add v, vp and the other two edges of H meeting v, creating a graph H' missing s. A similar argument holds if w is on $A(t,y]$. Since sp and pt are edges, these are the only two possible locations for w.

Case 4: There are at least two edges in G missing A, but all such edges meet another arc meeting y (or x, both cases are the same). We construct a new graph

H and choose a chord st as in Case 3. Now, if there is a vertex w on $A(s,p)$, the argument in Case 3 will not hold, because if $t = y$, all arcs from v in H' will go to the same arc of K'. If this does happen, then the argument in Case 3 shows that there cannot be a vertex on $A(p,t)$. Any chord meeting $A(s,p)$ must now have both vertices on $A[x,p]$, and the existence of such a chord then allows us to easily construct a graph H' missing a vertex between the vertices of the chord as in previous cases. It follows that there are no such chords, and thus any vertex on $A(s,p)$ is joined only to v. Such a vertex must be a 3-valent vertex in a t-triangle of G.

In every case we have reached a contradiction, thus we may add all of the edges of $G - K$, missing v, to K, while preserving 3^*-connectedness at each step. When all of these edges have been added, v is still a removable vertex. Adding any remaining edges in G meeting v does not change the removability of v, thus v is a removable vertex of G.

This completes the proof except for the case where every choice of G_0 contains all vertices of G, in which case the process described above cannot get started. In this case, we first consider a subgraph J of G, whose underlying graph is a wheel with a maximum number of spokes. Since G_0^* is a wheel, such a maximal wheel exists. If J does not contain all vertices of G, then a subgraph of J, that does not contain all vertices of G, can serve as a G_0.

Suppose that J^* has at least four spokes. If any path in J, corresponding to a spoke, has a vertex, then one can easily get a refinement of C_4 in J that misses that vertex. Thus J consists of a simple circuit C and a central vertex h joined to various vertices of C by edges.

If an edge of $G - J$ joins any two points on C, the reader may easily verify that there will be a refinement of G_0 missing a vertex. If no such edge exists then $J = G$, and G has t-triangles.

Suppose, now, that J is a refinement of C_4. Suppose there is an edge uv of G joining two open arcs A and B of J as shown in Fig. 2. If $B[u,y]$ is not an edge in G, then a refinement of C_4 is easily constructed missing the vertices on $B(u,y)$. By the same reasoning, $A[v,y]$ is an edge. If u is 3-valent in G, then G has a t-triangle. We leave it to the reader to check that if an edge is added from u then the resulting graph has a refinement of C_4 missing a vertex (note that an edge cannot join u to z or w, because then a larger wheel would exist).

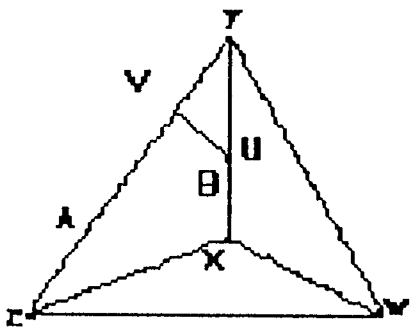

Fig. 2

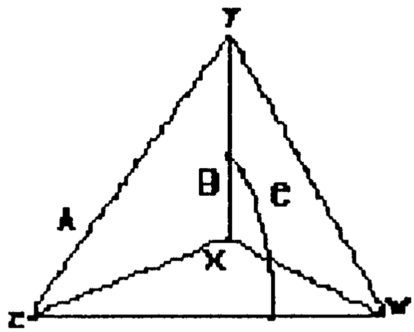

Fig. 3

Suppose that an edge e joins two open arcs A and B of J as shown in Fig. 3. This produces a refinement of $K_{3.3}$. It is easily checked that if one adds a vertex to any edge of $K_{3,3}$ then the resulting graph has a refinement of C_4 missing that vertex. It follows that J with e added is $K_{3,3}$, and that G is either $K_{3,3}$, or $K_{3,3}$ with edges added between vertices of $K_{3,3}$. It is easily checked that if even one new edge is added between vertices of $K_{3,3}$, then the resulting graph contains a refinement of C_4 missing at least one vertex. ∎

4. Generating 3-connected graphs

We shall say that G is obtained from H by **adding a t-edge**, provided H is obtained from G by removing an edge of a t-triangle opposite a 3-valent vertex, and coalescing edges at 2-valent vertices. The next theorem follows immediately from Theorem 1.

THEOREM 2: *The 3-connected graphs can be generated from C_4 and $K_{3,3}$ by adding vertices and adding t-edges.*

5. Other constructions

There are three other well known recursive constructions of the 3-connected graphs. In this section we show that they follow rather easily from our main theorem.

THEOREM 3: (Thomassen [2]) *Every 3-connected graph other than C_4 has a contractible edge.*

Proof: Let G be a 3-connected graph other than C_4. If G has a t-triangle, then it is easily seen that the edge meeting the triangle on its 3-valent vertex is contractible.

Suppose that G has no t-triangles. Let v be a removable vertex. If we take an edge vw meeting v and contract it, we obtain a graph that is G with v removed and edges added. As long as the edges added do not have both endpoints on the same edge of $G - v$, the result is a 3-connected graph. If one of the added edges, say wx, has both endpoints on an edge e of $G - v$ then these endpoints must be vertices of $G - v$, otherwise wxv is a t-triangle. If, however, there are added edges that make multiple edges with edges of $G - v$, then we coalesce the multiple edges into single edges, and the resulting graph is 3-connected.

The only case not covered by this argument is the case where G is $K_{3,3}$, for which the theorem is obviously true. ∎

THEOREM 4: (Tutte [3]) *The 3-connected graphs can be generated from the wheels by adding edges between existing vertices and splitting vertices so as not to produce triangles.*

Proof: Suppose that G is a 3-connected graph that is not a wheel. We shall say that an edge is t-contractible provided it is contractible and isn't an edge of a triangle. We shall say that it is t-removable if it is removable and it does not meet a 3-valent vertex. We therefore wish to show that G has either a t-contractible or a t-removable edge.

Case 1: G contains a t-triangle xa_1a_2 with vertex a_2 3-valent and joined to vertex a_0 by an edge. The edge xa_2 is removable, and is t-removable unless one of its edges is 3-valent. Also the edge a_0a_1 is contractible, and is t-contractible unless a_0a_1 lies on a triangle. Let us assume that xa_0a_1 is a triangle. If x is 3-valent, then G is C_4 thus we may assume that a_2 is 3-valent and is joined to a vertex a_3. If $a_3 = a_0$ then G is C_4 and we are done, so we assume that this is not the case. Now using the contractibility argument for a_2a_3 we conclude that xa_2a_3 is a triangle. Repeating these arguments, a_3 is 3-valent and is joined to a vertex a_4 which can't be the same as a_0 because then G would be a wheel. This argument can be continued indefinitely creating triangles meeting x with their other two vertices 3-valent. This contradicts the finiteness of the graph.

Case 2: G has no t-triangles. Let v be a removable vertex. As we have seen in Theorem 3, each edge meeting v is contractible, thus we are done unless each edge meeting v is an edge of a triangle. In this case v has valence at least 4, and it follows that each edge meeting v is removable. This is because removing an edge is the same as removing v then adding the vertex v back together with all but one of its original edges. If the resulting graph were not 3-connected, then all the added edges would meet one edge of $G - v$, and thus G would have a t-triangle. Now, however, a removable edge e meeting v must be t-removable, for otherwise e would meet a 3-valent vertex and lie on a triangle, a contradiction.

Again, the special case of $K_{3,3}$ is obvious. ∎

Similar arguments will also prove the well-known result that all 3-connected graphs except C_4 have removable edges.

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